

# Resit Exam Linear Algebra II

Friday 30/06/2023, 15:00–17:00

Exam Hall 1 C23 – G10

1 (2 + 2 + 2 + 2 = 8 pts)

Linear transformations, dimension theorem

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Consider the vector space  $P_2$  of real polynomials of degree  $\leq 2$ .

(a) Let  $T$  be the mapping from  $P_2$  to  $P_2$  defined by

$$(T(p))(x) := p(x) + x^2 p\left(\frac{1}{x}\right).$$

Show that  $T$  is a linear transformation.

(b) Find a matrix representation of  $T$  with respect to the basis  $\{1, x, 1 + 2x + x^2\}$ .

(c) Prove that the set

$$S = \{p \in P_2 \mid p(x) + x^2 p\left(\frac{1}{x}\right) = 0\}$$

is a subspace of  $P_2$ .

(d) Find the dimension of  $S$ .

2 (3 + 2 + 3 + 3 + 3 = 14 pts)

Inner products, adjoint

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Let  $V$  be an inner product space over  $\mathbb{C}$ , equipped with the inner product  $\langle \cdot, \cdot \rangle$ . Given an integer  $n \geq 2$  and a  $\mathbb{C}$ -linear transformation  $T: V \rightarrow V$  satisfying  $T^n = id$  (in other words,  $T^n(v) = v$  for every  $v \in V$ ). Define  $\langle\langle \cdot, \cdot \rangle\rangle: V \times V \rightarrow \mathbb{C}$  by

$$\langle\langle v, w \rangle\rangle := \sum_{k=1}^n \langle T^k(v), T^k(w) \rangle.$$

(a) Prove that  $\langle\langle \cdot, \cdot \rangle\rangle$  is an inner product on  $V$ .

(b) Show that  $\langle\langle T(v), T(w) \rangle\rangle = \langle\langle v, w \rangle\rangle$  for all  $v, w \in V$ .

(c) Explain why the adjoint of  $T$  with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  equals  $T^{n-1}$ .

(d) Show that if  $T(v) = \lambda v$  for some  $v \in V$  and  $\lambda \in \mathbb{C}$ , then  $\langle\langle v, v \rangle\rangle = n \langle v, v \rangle$ .

(e) Take complex numbers  $\lambda, \mu$  with  $\lambda \neq \mu$  and nonzero vectors  $v \in \text{Ker}(T - \lambda \cdot id)$  and  $w \in \text{Ker}(T - \mu \cdot id)$  (so by assumption  $\lambda$  and  $\mu$  are distinct eigenvalues of  $T$ ). Prove that  $v \perp w$  with respect to the inner product  $\langle\langle \cdot, \cdot \rangle\rangle$ .

**3** (2 + 3 + 1 + 2 + 3 + 3 = 14 pts) **Positive semidefiniteness, singular values**

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Consider a symmetric matrix  $M \in \mathbb{R}^{n \times n}$ .

- (a) Assume that  $M \geq 0$ , i.e.,  $M$  is *positive semidefinite*. Prove that all eigenvalues of  $M$  are non-negative.
- (b) Assume that all eigenvalues of  $M$  are non-negative. Prove that  $M \geq 0$ .  
*Hint:* write  $M = U\Lambda U^\top$  where  $U \in \mathbb{R}^{n \times n}$ , and  $\Lambda \in \mathbb{R}^{n \times n}$  is a diagonal matrix containing the eigenvalues of  $M$ .
- (c) Now consider a matrix  $P \in \mathbb{R}^{n \times n}$ . Show that  $P^\top P \geq 0$ .
- (d) Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be the eigenvalues of  $P^\top P$ . Consider a real number  $\gamma \geq 0$ . What are the eigenvalues of the matrix  $\gamma^2 I - P^\top P$ ?
- (e) During the lectures, we have seen that the *spectral norm*  $\|P\|_2$  equals the largest singular value  $\sigma_1$  of  $P$ . Use this fact to prove that if  $\gamma^2 I - P^\top P \geq 0$  then  $\|P\|_2 \leq \gamma$ .
- (f) Finally prove the converse to (e): if  $\|P\|_2 \leq \gamma$  then  $\gamma^2 I - P^\top P \geq 0$ .

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4 pts free